Fragility of Properness

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The theorem

In this short talk we show just a single theorem.

Theorem

Whenever $V \subsetneq W$ are models of ZFC with the same ordinals. Then there exists a poset $\mathbb{P}$ in V such that $\mathbb{P}$ is proper in V but improper in W.

Short background story

Karagila once asked the following question in his website, where he lists some open problems in set theory.

Problem (Karagila [1], Problem 3(b))

Is there a cardinal κ such that every proper poset remains proper after any κ-closed forcing?

This problem was solved negatively by the speaker [2] (2018).
After this, Karagila added another problem somewhat with an ‘opposite’ tone.

Problem (Karagila [1], Problem 5)

For any nontrivial poset $\mathbb{P}$, is there a proper poset $\mathbb{Q}$ whose properness is destroyed after forcing over $\mathbb{P}$?

Our theorem solves this problem affirmatively.
The background of these questions is explained more in detail in Karagila [3].

Proof of the theorem

First let:
$\kappa :=$ the least ordinal s.t. ${}^{\kappa}\mathrm{Ord} \cap (W \setminus V) \neq \emptyset$.
$\lambda :=$ the least ordinal s.t. ${}^{\kappa}\lambda \cap (W \setminus V) \neq \emptyset$.

Then we have:
κ is a regular infinite cardinal both in V and W.
λ is a cardinal in V with $\lambda \geq 2$.

We will prove the theorem by two cases.

Case 1 $\kappa > \omega$: We use a variation of Shelah's example ([4], XVII Observation 2.12, p.826) of two proper posets whose product is improper.
Case 2 $\kappa = \omega$: We generalize an argument by Shelah (introduced by Goldstern [5]) to prove some σ-closed forcing turns improper by adding a Cohen real.
Remark These techniques have been mentioned in Karagila [3] as 'examples'.

Case 1: $\kappa > \omega$ (1/3)

Note that $\omega_1^V = \omega_1^W$. (So we will write them as ω_1.)
Work in V.

Let $T :=$ the tree ${}^{<\kappa}\lambda$ (ordered by end-extension).
There are $\theta := \lambda^\kappa$ branches through T.
Note that by the choice of κ and λ, T has a branch in $W \setminus V$.

Let $\mathbb{P} := \mathrm{Add}(\omega)$ and $\dot{\mathbb{Q}} := \mathrm{Col}(\omega_1, \theta)^{V^{\mathbb{P}}}$.

$\mathbb{P} * \dot{\mathbb{Q}}$ adds no new branch through T by the following theorem, and thus in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ there are ω_1 branches through T.

Theorem (Mitchell [6])

Let $\mathrm{cf}\gamma > \omega$. Then forcing with $\mathrm{Add}(\omega) * (\sigma\text{-closed})$ adds no new sequence s of ordinals of length γ such that every initial segment of s is in the ground model.

Case 1: $\kappa > \omega$ (2/3)

Now work in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.

We can pick a club subset $\dot{C} \subseteq \kappa$ of order type ω_1.

Let $T \restriction \dot{C} := \{s \in T \mid \mathrm{length}(s) \in \dot{C}\}$.
Note that branches through $T \restriction \dot{C}$ are essentially those through T.
So $T \restriction \dot{C}$ is a tree of size and height ω_1 with ω_1 branches.

Now let $\dot{\mathbb{R}}$ be as in the following theorem.

Theorem (Baumgartner [7] §7)

Let $\mathcal{T}$ be a tree of size and height ω_1 with at most ω_1 branches. Then there is a c.c.c. poset $\mathbb{R}$ which specializes $\mathcal{T}$, that is,

$$\Vdash_{\mathbb{R}} \begin{array}{l}\text{“}\mathcal{T}\text{ is a countable union of subsets, each of which is}\\ \text{a union of mutually incompatible tails of branches}\\ \text{in the ground model.”}\end{array}$$

Case 1: $\kappa > \omega$ (3/3)

Note that (in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$) forcing with $\dot{\mathbb{R}}$ adds no new branches through $T \upharpoonright \dot{C}$.

Nor does any further extension, unless it collapses ω_1.

Now in V let $\tilde{\mathbb{P}} := \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$.

$\tilde{\mathbb{P}}$ is of the form (c.c.c.)*(σ-closed)*(c.c.c.) and thus proper in V.

Let G be any $\tilde{\mathbb{P}}$-generic over W (thus over V).

Then $W[G]$ is an extension of $V[G]$ and contains a branch through $T \upharpoonright \dot{C}$ which is not in V (and thus not in $V[G]$).

Therefore ω_1 is collapsed in $W[G]$, and therefore $\tilde{\mathbb{P}}$ is not proper.

Case 2: $\kappa = \omega$ (1/2)

In this case, we can prove the following:

Lemma

There exists $\mu > \omega_1^W$ regular in W such that $(\mathcal{P}_{\omega_1}\mu)^W \setminus V$ is stationary in W.

(Proof of Lemma) If $\lambda = 2$ (i.e. W has a real not in V), the conclusion holds for $\mu = \omega_2^W$, as proved by Gitik ([8], Theorem 1.1).

Otherwise, let $f \in {}^{\omega}\lambda \cap (W \setminus V)$.
Since $W \cap {}^{\omega}2 = V \cap {}^{\omega}2$, no countable set in V contains $\mathrm{ran}(f)$.
Now let μ be a W-regular cardinal such that $\mu \geq \omega_2^W$, λ.
Then

$$\{x \in (\mathcal{P}_{\omega_1}\mu)^W \mid \mathrm{ran}(f) \subseteq x\} \subseteq (\mathcal{P}_{\omega_1}\mu)^W \setminus V$$

contains a club subset of $(\mathcal{P}_{\omega_1}\mu)^W$. $\square$

Case 2: $\kappa = \omega$ (2/2)

Let μ be as in Lemma and let $\mathbb{P} := \mathrm{Col}(\omega_1, \mu)^V$. $\mathbb{P}$ is proper in V. Now work in W. Let θ be a sufficiently large regular cardinal. Then

$$Y = \{M \prec H_\theta \mid \mathbb{P} \in M, |M| = \omega, M \cap \mu \notin V\}$$

is stationary in $\mathcal{P}_{\omega_1} H_\theta$.
For each $M \in Y$, $M \cap \omega_1$ is an ordinal and so is $\delta := M \cap \omega_1^V$.
If $p \in \mathbb{P}$ were $(M, \mathbb{P})$-generic, by a density argument we would have

$$\mathrm{ran}(p \restriction \delta) = M \cap \mu \notin V,$$

which is absurd since $p \in V$.
Therefore $\mathbb{P}$ is not proper in W. $\square$

Question

<u>Question</u> For any two models $V \subsetneq W$ of ZFC with the same ordinals, can one find $\mathbb{P}$ totally proper in V but not in W?

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